Optimal Temperature Distribution on the Cascades of a Multistage Thermoelectric Module

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Abstract

The problem of finding the optimal sequence of temperature values on the cascades is expressed in terms of the system of linear equations. A multi-iteration search allows obtaining the solution with required accuracy. The method can be used for an arbitrary selection of thermoelectric (TE) materials on the cascades, taking into account temperature behavior of TE parameters. The method can allow for thermal losses on the TE module substrates.

The analysis of the solutions yielded by the offered method is given. It is shown that there exist a number of solutions close to the optimum. Each of them can be related to a new configuration of a TE module, which is very important for a designer's degrees of freedom.

Introduction

The major requirement for a TE module is to provide a given cooling capacity at a given temperature by the smallest possible power consumption and acceptable design. If the number of TE module stages and TE materials per stage are known, the basic task is to find the optimal sequence of temperature values on the cascades to ensure minimum of the heating coefficient or the maximum coefficient of performance (COP).

This problem was studied in many papers[1,2,3,4]. But it is only the Optimal Control Theory, based on the Pontriagin maximum principle [5], that can exhaust the problem, taking into account TE parameters temperature behavior. However the application of the Pontriagin maximum principle, if based on the kinetic equation [4], though describes a single TE element correctly, does *not* properly characterize the operation of a pellet in a TE module for the reason that a pellet ought to be described in terms of not absolute but of differential Seebeck coefficient. If instead of the kinetic equation the thermal conductance equation is applied, the Pontriagin principle transformations [5] can yield the correct results both for a TE element and a pellet but so far this work has not been consistently carried out.

A less sophisticated and nonetheless consistent method would make the problem physically more clear and easier to cope with. It would be also interesting to research the nearest vicinity of the optimum, which is not within the task of the Optimal Control approaches.

In practice it is often assumed that a multistage TE module maximum COP is provided when COP_i of the cascades are equal [1,2,6]. Certainly, while the TE module temperature difference (ΔT) is fixed, the less delta ΔT_i on the i-th stage, the higher COP_i on it but at the same time ΔT_i on the other stages grow and their COP_i drop. Therefore the case of equal COP_i is not bad but how *near* to the optimum is it? Papers [3,4] say this mode is not exactly optimal but the calculating models applied leave this statement unconvincing as the one [3] is too particular on $Z(T)$ and the other [4] is, though general on optimum, vague on its vicinity.

This paper faces the problem of finding an optimal temperature distribution along the cascades in a multistage TE module when no simplifying assumptions on $Z(T)$ [2,3] are induced. The aim of the paper is to offer an algorithm allowing to find a solution of the problem with required accuracy. We assume that the intermediate substrates thermal conductance is high enough to ignore the thermal losses on them.

Algorithm to Find Optimal Temperature Distribution on the Cascades

Consider an N-stage TE module, the cold and hot sides temperature given: T_0 and T_N. We numerate the cascades from the cold one (cascade 1) to the hot one (cascade N) – see Fig. 1.

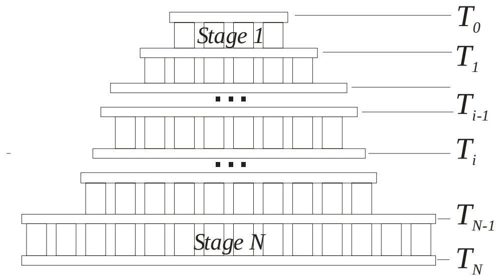

Figure 1: Stages and temperature values numeration

Introduce μ as the heating coefficient of the module and μ_i as that of the *i*-th cascade ($i=1,\ldots,N$). Assume that each cascade is characterized by its own figure-of-merit Z_i, ($i = 1, \ldots,N$) and the factor $M_i = \sqrt{1+Z_i \frac{T_{i-1}+T_i}{2}}$, ($i = 1,\ldots,N$). Further we consider Z_i constant per stage, given by a certain averaging procedure of the temperature dependent function $Z_i(T)$. This approach is reasonable as the consideration of the heat balance on a pellet ends allows expressing the equations in the traditional form via effective constant thermal parameters with different values on the cold and hot ends [7].

To find the optimal temperature sequence on the cascades it is necessary to minimize the module heating coefficient μ , i.e. to solve the equations system:

$$\frac{\partial \mu}{\partial T_i} = 0 \qquad (i = 1,...,N-1) \tag{1}$$

It is evident that μ is minimized when all the cascades operate at the maximum COP$_i$ (minimum μ_i). Therefore μ_i can be written as:

$$\mu_i = \frac{M_i T_i - T_{i-1}}{M_i T_{i-1} - T_i} \qquad (i = 1,...,N) \tag{2}$$

Then the system (1) can be transformed to:

$$\frac{\partial \mu_i}{\partial T_i} / \frac{\partial \mu_{i+1}}{\partial T_i} + \frac{\mu_i}{\mu_{i+1}} = 0 \qquad (i = 1,...,N-1) \tag{3}$$

One has treated this system not once but always confining oneself to a model temperature dependence of M_i [2,3]. For unrestricted values of Z_i the solution has never been sought for. Our objective is to fill the gap.

Keeping in mind that the averaging procedure makes Z_i a function of temperature $Z_i = Z_i(T_i, T_{i-1})$ we can rewrite system (3) in the following form $(i = 1,...,N-1)$:

$$\frac{M_{i+1}T_{i+1} - T_i}{\left(M_i T_{i-1} - T_i\right)}\left[\left(M_i^2 - 1\right)T_{i-1} - \frac{\partial M_i}{\partial T_i}\left(T_i^2 - T_{i-1}^2\right)\right] - \frac{M_i T_i - T_{i-1}}{\left(M_{i+1} T_i - T_{i+1}\right)}\left[\left(M_{i+1}^2 - 1\right)T_{i+1} + \frac{\partial M_{i+1}}{\partial T_i}\left(T_{i+1}^2 - T_i^2\right)\right] = 0 \tag{4}$$

Suppose we have a solution $T_i^{(0)}$ of Eqs. (4) close to the true solution. Then the true T_i have the following form:

$$T_i = T_i^{(0)} + \delta_i, \tag{5}$$

where δ_i is a small value.

Following Eq. (5) the expressions for Z_i and M_i can be written in the form:

$$Z_i = Z_i^{(0)} + \delta Z_i, \; M_i = M_i^{(0)} + \delta M_i \tag{6}$$

Here $Z_i^{(0)}$ and $M_i^{(0)}$ correspond to $T_i^{(0)}$ and δZ_i, δM_i are small corrections when proceeding to T_i.

Expanding Z_i, M_i into the series about $T_i^{(0)}$ and taking the first-order derivatives only, we obtain the following:

$$\delta Z_i = \frac{\partial Z_i^{(0)}}{\partial T_i}\delta_i + \frac{\partial Z_i^{(0)}}{\partial T_{i-1}}\delta_{i-1}, \tag{7}$$

$$\delta M_i = \frac{\partial M_i^{(0)}}{\partial T_i}\delta_i + \frac{\partial M_i^{(0)}}{\partial T_{i-1}}\delta_{i-1}$$

Within this designation we transform Eqs. (4) into the following $(i = 1,...,N-1)$:

$$\frac{\left(B_{i+1}^{(0)} + M_{i+1}^{(0)}\delta_{i+1} + T_{i+1}^{(0)}\delta M_{i+1} - \delta_i\right)\left(A_{i+1}^{(0)} + M_{i+1}^{(0)}\delta_i + T_i^{(0)}\delta M_{i+1} - \delta_{i+1}\right)}{\left(A_i^{(0)} + M_i^{(0)}\delta_{i-1} + T_{i-1}^{(0)}\delta M_i - \delta_i\right)\left(B_i^{(0)} + M_i^{(0)}\delta_i + T_i^{(0)}\delta M_i - \delta_{i-1}\right)} \times \frac{\left[C_i^{(0)} + 2M_i^{(0)}T_{i-1}^{(0)}\delta M_i + \left(M_i^{(0)2} - 1\right)\delta_{i-1} - \frac{2\partial M_i}{\partial T_i}\left(T_i^{(0)}\delta_i - T_{i-1}^{(0)}\delta_{i-1}\right)\right]}{\left[D_{i+1}^{(0)} + 2M_{i+1}^{(0)}T_{i+1}^{(0)}\delta M_{i+1} + \left(M_{i+1}^{(0)2} - 1\right)\delta_{i+1} + \frac{2\partial M_{i+1}}{\partial T_i}\left(T_{i+1}^{(0)}\delta_{i+1} - T_i^{(0)}\delta_i\right)\right]} = 1 \tag{8}$$

In Eqs. (8) we used the notations:

$$A_i^{(0)} = M_i^{(0)}T_{i-1}^{(0)} - T_i^{(0)},$$

$$B_i^{(0)} = M_i^{(0)}T_i^{(0)} - T_{i-1}^{(0)},$$

$$C_i^{(0)} = \left(\left(M_i^{(0)}\right)^2 - 1\right)T_{i-1}^{(0)} - \frac{\partial M_i^{(0)}}{\partial T_i}\left[\left(T_i^{(0)}\right)^2 - \left(T_{i-1}^{(0)}\right)^2\right], \tag{9}$$

$$D_{i+1}^{(0)} = \left(\left(M_{i+1}^{(0)}\right)^2 - 1\right)T_{i+1}^{(0)} + \frac{\partial M_{i+1}}{\partial T_i}\left[\left(T_{i+1}^{(0)}\right)^2 - \left(T_i^{(0)}\right)^2\right]$$

Expanding the terms in the parentheses of the denominator in Eqs. (8) into the series of δ_i and taking the linear ones, we come to the following equations linear in δ_i:

$$a_{i(i-1)}\delta_{i-1} + a_{ii}\delta_i + a_{i(i+1)}\delta_{I+1} = b_i, \quad (i = 1,...,N-1) \tag{10}$$

In system (10) we set $\delta_0 = 0$, $\delta_N = 0$, as the corresponding temperature values are fixed. The left part coefficients in Eqs. (10) compose a triagonal matrix.

The coefficients in Eqs. (10) are expressed as (11)-(13):

$$a_{i(i-1)} = \frac{1}{B_i^{(0)}} - \frac{M_i^{(0)}}{A_i^{(0)}} + \frac{\left(M_i^{(0)2} - 1\right)}{C_i^{(0)}} + \frac{\frac{2\partial M_i^{(0)}}{\partial T_i}T_{i-1}^{(0)}}{C_i^{(0)}} - \frac{\partial M_i^{(0)}}{\partial T_{i-1}}\left(\frac{T_{i-1}^{(0)}}{A_i^{(0)}} + \frac{T_i^{(0)}}{B_i^{(0)}} - \frac{2M_i^{(0)}T_{i-1}^{(0)}}{C_i^{(0)}}\right) \tag{11}$$

$$a_{i(i+1)} = \frac{M_{i+1}^{(0)}}{B_{i+1}^{(0)}} - \frac{1}{A_{i+1}^{(0)}} - \frac{\left(M_{i+1}^{(0)2} - 1\right)}{D_{i+1}^{(0)}} - \frac{2\partial M_{i+1}^{(0)}}{\partial T_i}\frac{T_{i+1}^{(0)}}{D_{i+1}^{(0)}} + \frac{\partial M_{i+1}^{(0)}}{\partial T_{i+1}}\left(\frac{T_{i+1}^{(0)}}{B_{i+1}^{(0)}} + \frac{T_i^{(0)}}{A_{i+1}^{(0)}} - \frac{2M_{i+1}^{(0)}T_{i+1}^{(0)}}{D_{i+1}^{(0)}}\right) \tag{12}$$

$$a_{ii} = -\frac{1}{B_{i+1}^{(0)}} + \frac{M_{i+1}^{(0)}}{A_{i+1}^{(0)}} + \frac{1}{A_i^{(0)}} - \frac{M_i^{(0)}}{B_i^{(0)}} - \frac{2\partial M_i}{\partial T_i}\frac{T_i^{(0)}}{C_i^{(0)}} +$$
$$+ \frac{2\partial M_{i+1}^{(0)}}{\partial T_i}\frac{T_i^{(0)}}{D_{i+1}} + \frac{\partial M_{i+1}^{(0)}}{\partial T_i}\left(\frac{T_{i+1}^{(0)}}{B_{i+1}^{(0)}} + \frac{T_i^{(0)}}{A_{i+1}^{(0)}} - \frac{2M_{i+1}^{(0)}T_{i+1}^{(0)}}{D_{i+1}^{(0)}}\right) - \quad (13)$$
$$- \frac{\partial M_i^{(0)-}}{\partial T_i}\left(\frac{T_{i-1}^{(0)}}{A_i^{(0)}} + \frac{T_i^{(0)}}{B_i^{(0)}} - \frac{2M_i^{(0)}T_{i-1}^{(0)}}{C_i^{(0)}}\right)$$

Eqs. (10), though may seem cumbersome, can be easily solved with the help of the standard programming tools. It is convenient to take a zero approach $T_i^{(0)}$ for the case M_i is independent of the cascade number [2]: $T_i^{(0)} = T_0(T_N / T_0)^{i/N}$, $i=0,1,\ldots N$.

After solving Eqs. (10) and finding δ_i we calculate T_i by Eq. (5) and take the found values as the zero approach for the next step, etc. The process converges fast enough. The values Z_i, M_i are necessary to correct at each step.

The obtained solution is not restricted by any assumptions on Z_i, M_i.

In the suggested method we can consider corrections Δ for thermal losses on TE module substrates. Then for the i-th cascade the hot side temperature should be T_i-$\Delta/2$ and the cold side temperature should be $T_{i-1}+\Delta/2$.

Below we apply the method to calculating the optimal temperature distribution on a 5-stage TE module cascades.

An Example of the Method in Work

The temperature distribution is calculated for a 5-stage TE module cooling from 308 K down to 175 K. TE materials are taken with a higher value of the Seebeck coefficient at room temperature for smaller numbers of the cascades.

As effective Z_i in the calculations we may take that averaged over the values on the ends of the i-th stage:

$$Zi = \frac{Z(T_i) + Z(T_{i-1})}{2} \quad (14)$$

In this case the derivatives are equal to:

$$\frac{\partial Z_i}{\partial T_i} = \frac{\left.\frac{\partial Z_i}{\partial T}\right|_{T=T_i}}{2} \text{ and } \frac{\partial Z_i}{\partial T_{i-1}} = \frac{\left.\frac{\partial Z_i}{\partial T}\right|_{T=T_{i-1}}}{2} \quad (15)$$

Here we assume that the Thomson effect makes more realistic effective Z at the cascade hot sides [7], the corresponding values are:

$$Z_i = Z(T_i),\ \frac{\partial Z_i}{\partial T_i} = \left.\frac{\partial Z_i}{\partial T}\right|_{T=T_i},\ \frac{\partial Z_i}{\partial T_{i-1}} = 0\ . \quad (16)$$

The results of the calculations are given in Table 1. It shows that even the zero-approach gives satisfying results (4% worse than in the optimum case), so from the very start of the iteration δ_i are small enough for a fast converging. The δ_i -values for the given data are not higher than 10^{-2} K.

Table 1: *The temperature distribution on the 5-stage TE module cascades*

i	Zero-approach			Equal COP$_i$			Optimum		
	T_i, K	μ_i	M_i	T, K	μ_i	M_i	T_i, K	μ_i	M_i
0	175			175			175		
1	195.9	4.587	1.193	191.2	3.120	1.188	191.9	3.289	1.190
2	219.4	3.528	1.225	211.4	3.121	1.216	213.1	3.280	1.218
3	245.7	3.031	1.252	236.0	3.121	1.239	237.7	3.059	1.242
4	275.1	2.608	1.290	267.8	3.111	1.281	269.5	3.055	1.283
5	308	2.397	1.318	308	3.111	1.315	308	2.912	1.316
	μ = 306.62			μ = 294.16			μ = 293.51		

We see that the method based on equal COP$_i$ and the method developed here (optimum cooling mode) yield the results very close in heating coeefficients μ, though the difference in temperature distribution and μ_i is quite noticeable. At the next step of the optimal module simulation it is bound to influence the module configuration. It may be shown that in the case of equal COP$_i$ the cascading coefficients (the ratio of pellets numbers of the neighboring cascades) fluctuate near the same value. In the optimum mode the cascading coefficients tend to be smaller on the bottom (hotter) stages. In this mode the pyramide of a series-connected TE module expands to the bottom not so equal-rate for the heating coefficients on the upper stages exceed those on the lower ones. That may be gained by pellets number variation, advantageous geometry of pellets and optimal electric current.

Discussions and Conclusions

The data obtained (see Table 1) confirm the conclusion [3] that the traditional method of equal COP$_i$ gives good results not because it is correct but because the temperature dependences of the figure-of-merit in traditional chalcogenides of Bi-Sb are such that the solution happens to fall near the true extremum, which is very flat. However in our algorithm it is not necessary to restrict *M(T)* [2,3] to one power function, which may not be true for all cascades.

Let us investigate the vicinity of the optimum. Suppose $T_{1min} - T_{4min}$ is a sequence of optimal temperature values for the optimum cooling mode, and $T_{1e} - T_{4e}$ is that for the equal COP$_i$. Let us imagine we are moving along the parametric curve across the points given by these T_i:

$$T_i = T_{i\,e} - (T_{i\,min} - T_{i\,e})\tau\ , \quad (i = 1,..,4) \quad (17)$$

where τ is a parameter.

At $\tau = -1$ the values of temperature T_i correspond to the optimum cooling mode and at $\tau = 0$ they relate to the equal COP$_i$ cooling mode. While moving along this parametric trajectory (depicted in Fig. 2) we obtain the function $\mu(T_1, T_2, T_3, T_4)$ in a range around the minimum

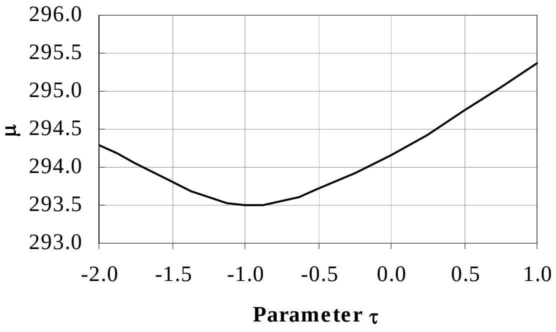

Figure 2: *Dependence of the heating coefficient on the parameter* τ

We see that the heating coefficient minimum is captured by $\tau = -1$, and in the direction opposite the equal COP_i ($\tau = 0$) at the same distance $\Delta\tau = 1$ from the optimum we intercept nearly the same μ-value. However at $\tau = -2$ the distributions of temperature and heating coefficients on the cascades (and, consequently, the module geometry) are quite different and are given in Table 2.

Table 2: *The temperature values and heating coefficients sequence on the cascades of the 5-stage module at* $\tau = -2$.

i	T_i , K	μ_i	M_i
1	192.58	3.4684	1.1896
2	214.70	3.4408	1.2193
3	239.38	2.9998	1.2439
4	271.24	3.0058	1.2847
5	308	2.7349	1.3165

The overall heating coefficient is equal to 294.29, which is very close to that in the mode of equal COP_i (see Table 1).

Therefore this method of finding true optimum of a multistage TE module operation shows that there are many solutions for the temperature distributions in its nearest vicinity. These distributions provide the heating coefficient quite close to the optimal. The case of equal COP_i is one of them. Each of the solutions is related to a specific module cofiguration (number of pellets, pellets geometry per stage) and optimal parameters. That is why within slight deviations from the optimum it is possible to select a module design to meet TE module requirements.

The suggested method refers to the case when cascading coefficients are relatively moderate (around 3, which is normal for commercial TE modules), and when the temperature on the substrate can be considered constant.

The method is welcome for optimal TE modules engineering.

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